

LINEAR ORTHOGONALITY PRESERVERS OF HILBERT BUNDLES

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ABSTRACT. Due to the corresponding fact concerning Hilbert spaces, it is natural to ask if the linearity and the orthogonality structure of a Hilbert C^* -module determine its C^* -algebra-valued inner product. We verify this in the case when the C^* -algebra is commutative (or equivalently, we consider a Hilbert bundle over a locally compact Hausdorff space). More precisely, a \mathbb{C} -linear map θ (not assumed to be bounded) between two Hilbert C^* -modules is said to be “orthogonality preserving” if $\langle \theta(x), \theta(y) \rangle = 0$ whenever $\langle x, y \rangle = 0$. We prove that if θ is an orthogonality preserving map from a full Hilbert $C_0(\Omega)$ -module E into another Hilbert $C_0(\Omega)$ -module F that satisfies a weaker notion of $C_0(\Omega)$ -linearity (known as “localness”), then θ is bounded and there exists $\phi \in C_b(\Omega)_+$ such that

$$\langle \theta(x), \theta(y) \rangle = \phi \cdot \langle x, y \rangle, \quad \forall x, y \in E.$$

On the other hand, if F is a full Hilbert C^* -module over another commutative C^* -algebra $C_0(\Delta)$, we show that a “bi-orthogonality preserving” bijective map θ with some “local-type property” will be bounded and satisfy

$$\langle \theta(x), \theta(y) \rangle = \phi \cdot \langle x, y \rangle \circ \sigma, \quad \forall x, y \in E$$

where $\phi \in C_b(\Omega)_+$ and $\sigma : \Delta \rightarrow \Omega$ is a homeomorphism.

1. INTRODUCTION

It is a common knowledge that the inner product of a Hilbert space determines both the norm and the orthogonality structures; and conversely, the norm structure determines the inner product structure. It might be a bit less well-known that the orthogonality structure of a Hilbert space also determines its norm structure. Indeed, if θ is a linear map between Hilbert spaces preserving orthogonality, then it is easy to see that θ is a scalar multiple of an isometry (see [5, 6]).

We are interested in the corresponding relations for Hilbert C^* -modules. Note that in the case of a commutative C^* -algebra $C_0(\Omega)$, Hilbert $C_0(\Omega)$ -modules are the same as Hilbert bundles, or equivalently, continuous fields of Hilbert spaces over Ω . By modifying the proof of [12, Theorem 6] (see also [13, 16, 9]), one can show that any surjective isometry between two continuous fields of Hilbert spaces with non-zero fibres over each

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point is given by a homeomorphism and a field of unitaries. Thus, the norm structure (and linearity) determines the unitary structure in this situation.

Our primary concern is the question of whether the orthogonality structure of a Hilbert C^* -module determines its unitary structure. More precisely, let A be a C^* -algebra, and E and F be two Hilbert A -modules. If $\theta : E \rightarrow F$ is an A -module homomorphism, which is not assumed to be bounded but preserves orthogonality (that is, $\langle \theta(x), \theta(y) \rangle_A = 0$ whenever $\langle x, y \rangle_A = 0$), we ask whether there is a central positive multiplier u in $M(A)$ such that

$$\langle \theta(e), \theta(f) \rangle_A = u \langle e, f \rangle_A, \quad \forall e, f \in E.$$

When $A = \mathbb{C}$, it reduces to the case of Hilbert spaces. Recently, D. Ilišević and A. Turnšek [10] gave a positive answer in the case when A is a standard C^* -algebra (that is, $\mathcal{K}(H) \subseteq A \subseteq \mathcal{L}(H)$).

In this article, we will give a positive answer when A is a commutative C^* -algebra (actually, we prove a slightly stronger result that replaces the A -linearity with the localness property; see Definition 2.1). On the other hand, we will also consider bijective bi-orthogonality preserving maps between Hilbert C^* -modules over different commutative C^* -algebras. We show that if such a map also satisfies certain local-type property (see Definition 3.10) but not assumed to be bounded, then it is given by a homeomorphism (between the base spaces) and a “continuous field of unitaries”. We remark that in this case of Hilbert C^* -modules over different commutative C^* -algebras, one cannot define “ A -linearity” but have to consider localness property. This is one of the reasons for considering local maps. We remark also that this case does not cover the case of Hilbert C^* -modules over the same commutative C^* -algebra because we need to assume that the map is both bijective and bi-orthogonality preserving.

Note that if Ω is a locally compact Hausdorff space and H is a Hilbert space, then $C_0(\Omega, H)$ is a Hilbert $C_0(\Omega)$ -module. As far as we know, even in this case our results are new, and the technique in the proofs are non-standard and non-trivial comparing with those in the literatures [1, 4, 8, 11], concerning separating or zero-product preservers (although some statements look similar). In a forthcoming paper of the authors, we will study the case when the underlying C^* -algebra is not commutative.

2. TERMINOLOGIES AND NOTATIONS

Recall that a (right) *Hilbert C^* -module* E over a C^* -algebra A is a right A -module equipped with an A -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ such that the following conditions hold for all $x, y \in E$ and all $a \in A$.

$$(1) \quad \langle x, ya \rangle = \langle x, y \rangle a.$$

- (2) $\langle x, y \rangle^* = \langle y, x \rangle$.
- (3) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ exactly when $x = 0$.

Moreover, E is a Banach space equipped with the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We also call E a *Hilbert A -module* in this case. A complex linear map $\theta : E \rightarrow F$ between two Hilbert A -modules is called an *A -module homomorphism* if $\theta(xa) = \theta(x)a$ for all $a \in A$ and $x \in E$. See, for example, [15] or [20], for a general introduction to the theory of Hilbert C^* -modules. In this paper, we are interested in the case when the underlying C^* -algebra A is abelian, that is, $A = C_0(\Omega)$ consisting of all continuous complex-valued functions defined on a locally compact Hausdorff space Ω vanishing at infinity.

Definition 2.1. Let A be a C^* -algebra. Suppose that E and F are Hilbert A -modules. A \mathbb{C} -linear map $\theta : E \rightarrow F$ is said to be *local* if $\theta(e)a = 0$ whenever $ea = 0$ for any $e \in E$ and $a \in A$.

The idea of local linear maps is found in many researches in analysis. For example, a theorem of Peetre [19] states that local linear maps of the space of smooth functions defined on a manifold modeled on \mathbb{R}^n are exactly linear differential operators (see [18]). This is further extended to the case of vector-valued differentiable functions defined on a finite dimensional manifold by Kantrowitz and Neumann [14] and Araujo [3], and in the Banach $C^1[0, 1]$ -module setting by Alaminos et. al. [2]. Note that every A -module homomorphism is local. Conversely, every *bounded* local map is an A -module homomorphism ([17, Proposition A.1]). See Remark 3.4 below for more information.

Notation 2.2. Throughout this article, Ω and Δ are two locally compact Hausdorff spaces, and Ω_∞ is the one-point compactification of Ω . Moreover, E and F are respectively, a (right) Hilbert $C_0(\Omega)$ -module and a (right) Hilbert $C_0(\Delta)$ -module, while $\theta : E \rightarrow F$ is a \mathbb{C} -linear map (not assumed to be bounded). We denote by $\mathcal{B}_{C_0(\Omega)}(E, F)$ the set of all bounded $C_0(\Omega)$ -module homomorphisms from E into F . For any $\omega \in \Omega$, we let $\mathcal{N}_\Omega(\omega)$ be the set of all compact neighborhoods of ω in Ω . If $S \subseteq \Omega$, we denote by $\text{Int}_\Omega(S)$ the interior of S in Ω . Moreover, if $U, V \subseteq \Omega$ such that the closure of V is a compact subset of $\text{Int}_\Omega(U)$, we denote by $\mathcal{U}_\Omega(V, U)$ the collection of all $\lambda \in C_0(\Omega)$ with $0 \leq \lambda \leq 1$, $\lambda \equiv 1$ on V and λ vanishes outside U .

Note that any Hilbert $C_0(\Omega)$ -module E can be regarded as a Hilbert $C(\Omega_\infty)$ -module, and the results in [7] can be applied. In particular, E is the space of C_0 -sections (that is, continuous sections that vanish at infinity) of an (F)-Hilbert bundle Ξ^E over Ω_∞ (see [7, p. 49]).

We define $|f|(\omega) := \|f(\omega)\|$ for all $f \in E$ and $\omega \in \Omega$. For any closed subset $S \subseteq \Omega_\infty$ and $\omega \in \Omega_\infty$, we set

$$K_S^E := \{f \in E : f(\omega) = 0, \forall \omega \in S\} \quad \text{and} \quad I_\omega := \bigcup_{V \in \mathcal{N}_{\Omega_\infty}(\omega)} K_V^E$$

(for simplicity, we also denote $K_\omega^E := K_{\{\omega\}}^E$). Notice that $K_\infty^E = E$ and the fibre of Ξ^E at $\omega \in \Omega_\infty$ is given by $\Xi_\omega^E = E/K_\omega^E$. Furthermore, K_S^E is a Hilbert $K_S^{C_0(\Omega)}$ -module and $K_S^E = E \cdot \overline{K_S^{C_0(\Omega)}}$.

On the other hand, we denote

$$\Delta_\theta := \{\nu \in \Delta : \theta(E) \not\subseteq K_\nu^F\} = \{\nu \in \Delta : \theta(e)(\nu) \neq 0 \text{ for some } e \in E\}.$$

Then Δ_θ is an open subset of Δ , and we put

$$\Omega_E := \{\omega \in \Omega : \Xi_\omega^E \neq (0)\}.$$

Let $\Omega_0 \subseteq \Omega$ be an open set. As in [7, p. 10], we denote by $\Xi^E|_{\Omega_0}$ the restriction of Ξ^E to Ω_0 and by E_{Ω_0} the set of C_0 -sections on $\Xi^E|_{\Omega_0}$. One can identify

$$C_0(\Omega_0) = K_{\Omega \setminus \Omega_0}^{C_0(\Omega)} \quad \text{and} \quad E_{\Omega_0} = K_{\Omega \setminus \Omega_0}^E.$$

3. ORTHOGONALITY PRESERVING MAPS BETWEEN HILBERT $C_0(\Omega)$ -MODULES

Let us first recall the following two technical lemmas from [17, Lemmas 3.1 and 3.3, and Theorem 3.7] (see also [17, Remark 3.4]), which summarize, unify, and generalize techniques sporadically used in the literatures [4, 8, 11].

Lemma 3.1. *If $\sigma : \Delta_\theta \rightarrow \Omega_\infty$ is a map satisfying $\theta(I_{\sigma(\nu)}^E) \subseteq K_\nu^F$ (for any $\nu \in \Delta_\theta$), then σ is continuous.*

Lemma 3.2. *Let $\sigma : \Delta \rightarrow \Omega$ be a map (not assumed to be continuous) such that $\theta(I_{\sigma(\nu)}^E) \subseteq K_\nu^F$ for every $\nu \in \Delta$.*

(a) *If $\mathfrak{U}_\theta := \{\nu \in \Delta : \sup_{\|e\| \leq 1} \|\theta(e)(\nu)\| = \infty\}$, then $\sigma(\mathfrak{U}_\theta)$ is a finite set.*

(b) *If $\mathfrak{N}_{\theta, \sigma} := \{\nu \in \Delta : \theta(K_{\sigma(\nu)}^E) \not\subseteq K_\nu^F\}$, then $\mathfrak{N}_{\theta, \sigma} \subseteq \mathfrak{U}_\theta$ and $\sigma(\mathfrak{N}_{\theta, \sigma})$ consists of non-isolated points in Ω .*

(c) *If σ is injective and sends isolated points in Δ to isolated points in Ω , then $\mathfrak{N}_{\theta, \sigma} = \emptyset$ and there exists a finite set T consisting of isolated points of Δ , a bounded linear map $\theta_0 : K_{\sigma(T)}^E \rightarrow K_T^F$ as well as linear maps $\theta_\nu : \Xi_{\sigma(\nu)}^E \rightarrow \Xi_\nu^F$ for all $\nu \in T$, such that $E = K_{\sigma(T)}^E \oplus \bigoplus_{\nu \in T} \Xi_{\sigma(\nu)}^E$,*

$$F = K_T^F \oplus \bigoplus_{\nu \in T} \Xi_\nu^F \quad \text{and} \quad \theta = \theta_0 \oplus \bigoplus_{\nu \in T} \theta_\nu.$$

For any $\nu \in \Delta \setminus \mathfrak{N}_{\theta, \sigma}$, one can define $\theta_\nu : \Xi_{\sigma(\nu)}^E \rightarrow \Xi_\nu^F$ by

$$(3.1) \quad \theta_\nu(e + K_{\sigma(\nu)}^E) = \theta(e) + K_\nu^F, \quad \forall e \in E,$$

or equivalently, $\theta_\nu(e(\sigma(\nu))) = (\theta(e))(\nu)$ for all $e \in E$.

Lemma 3.3. *Let σ and \mathfrak{U}_θ be the same as in Lemma 3.2. Suppose, in addition, that σ is injective and θ is orthogonality preserving. Then there exists a bounded function $\psi : \Delta \setminus \mathfrak{U}_\theta \rightarrow \mathbb{R}_+$ such that*

$$(3.2) \quad \langle \theta(e), \theta(g) \rangle(\nu) = \psi(\nu)^2 \langle e, g \rangle(\sigma(\nu)), \quad \forall e, g \in E, \forall \nu \in \Delta \setminus \mathfrak{U}_\theta.$$

Moreover, for each $\nu \in \Delta_\theta$, there is an isometry $\iota_\nu : \Xi_{\sigma(\nu)}^E \rightarrow \Xi_\nu^F$ such that

$$\theta(e)(\nu) = \psi(\nu) \iota_\nu(e(\sigma(\nu))), \quad \forall e \in E, \forall \nu \in \Delta_\theta \setminus \mathfrak{U}_\theta.$$

Proof. Fix any $\nu \in \Delta_\theta \setminus \mathfrak{U}_\theta$. By Lemma 3.2(b), the map θ_ν as in (3.1) is well-defined. Suppose that η_1 and η_2 are orthogonal elements in $\Xi_{\sigma(\nu)}^E$ with $\eta_1 \neq 0$ (it is possible because $\Delta_\theta \setminus \mathfrak{N}_{\theta, \sigma} \subseteq \sigma^{-1}(\Omega_E)$), and $g_1, g_2 \in E$ with $g_i(\sigma(\nu)) = \eta_i$ for $i = 1, 2$. If $V \in \mathcal{N}_\Omega(\sigma(\nu))$ is such that g_1 is non-vanishing on V , then by replacing g_2 with

$$\left(g_2 - \frac{\langle g_2, g_1 \rangle}{|g_1|^2} g_1 \right) \lambda$$

where $\lambda \in \mathcal{U}_\Omega(\{\sigma(\nu)\}, V)$, we see that there are orthogonal elements $e_1, e_2 \in E$ with $e_i(\sigma(\nu)) = \eta_i$ for $i = 1, 2$. Hence, θ_ν is non-zero (because $\nu \in \Delta_\theta$) and is an orthogonality preserving \mathbb{C} -linear map between Hilbert spaces. Consequently, there exist an isometry $\iota_\nu : \Xi_{\sigma(\nu)}^E \rightarrow \Xi_\nu^F$ and a unique scalar $\psi(\nu) > 0$ such that $\theta_\nu = \psi(\nu) \iota_\nu$. For any $\nu \in \Delta \setminus \Delta_\theta$, we set $\psi(\nu) = 0$. Then clearly (3.2) holds. Next, we show that ψ is a bounded function on $\Delta \setminus \mathfrak{U}_\theta$. Suppose that it is not the case. Then there exist distinct points $\nu_n \in \Delta_\theta \setminus \mathfrak{U}_\theta$ such that $\psi(\nu_n) > n^3$. If $e_n \in E$ with $\|e_n\| = 1$ and the modular function $|e_n|(\sigma(\nu_n)) = \sqrt{\langle e_n, e_n \rangle(\sigma(\nu_n))} \geq (n-1)/n$ (note that $\nu_n \in \sigma^{-1}(\Omega_E)$), then because of (3.2),

$$|\theta(e_n)|(\nu_n) = \psi(\nu_n) |e_n|(\sigma(\nu_n)) > n^2(n-1).$$

As $\{\sigma(\nu_n)\}$ is a set of distinct points (note that σ is injective), by passing to a subsequence if necessary, we can assume that there are $U_n \in \mathcal{N}_\Omega(\sigma(\nu_n))$ such that $U_n \cap U_m = \emptyset$ when $m \neq n$. Now, pick any $V_n \in \mathcal{N}_\Omega(\sigma(\nu_n))$ with $V_n \subseteq \text{Int}_\Omega(U_n)$ and choose a function $\lambda_n \in \mathcal{U}_\Omega(V_n, U_n)$ for all $n \in \mathbb{N}$. Define $e := \sum_{k=1}^\infty \frac{e_k \lambda_k^2}{k^2} \in E$. For any $n \in \mathbb{N}$, as $n^2 e - e_n \lambda_n^2 \in K_{U_n}^E$ and $e_n - e_n \lambda_n^2 = e_n(1 - \lambda_n^2) \in K_{V_n}^E$, we have

$$\|\theta(e)\| \geq \|\theta(e)(\nu_n)\| = \frac{\|\theta(e_n \lambda_n^2)(\nu_n)\|}{n^2} = \frac{\|\theta(e_n)(\nu_n)\|}{n^2} > n-1$$

(by the relation between θ and σ) which is a contradiction. \square

3.1. Hilbert bundles over the same base space.

Remark 3.4. For any $e \in E$, we denote

$$\text{supp}_\Omega e := \overline{\{\omega \in \Omega : e(\omega) \neq 0\}}.$$

It is not hard to check that the following statements are equivalent (which tells us that local maps are the same as *support shrinking maps* [8]):

- (i) θ is local (see Definition 2.1);
- (ii) $\theta(K_V^E) \subseteq K_V^F$ for any non-empty open set V ;
- (iii) $\text{supp}_\Omega \theta(e) \subseteq \text{supp}_\Omega e$ for every $e \in E$;
- (iv) $\text{supp}_\Omega \theta(e)\lambda \subseteq \text{supp}_\Omega e$ for each $e \in E$ and $\lambda \in C_0(\Omega)$.

Theorem 3.5. *Let Ω be a locally compact Hausdorff space, and let E and F be two Hilbert $C_0(\Omega)$ -modules. Suppose that $\theta : E \rightarrow F$ is an orthogonality preserving local \mathbb{C} -linear map. The following assertions hold.*

(a) $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$.

(b) *There is a bounded non-negative function φ on Ω which is continuous on Ω_E such that*

$$\langle \theta(e), \theta(g) \rangle = \varphi \cdot \langle e, g \rangle, \quad \forall e, g \in E.$$

(c) *There exist a strictly positive element $\psi_0 \in C_b(\Omega_\theta)_+$ and $J \in \mathcal{B}_{C_0(\Omega_\theta)}(E_{\Omega_\theta}, F_{\Omega_\theta})$ such that the fiber map J_ω is an isometry for each $\omega \in \Omega_\theta$ and*

$$\theta(e)(\omega) = \psi_0(\omega)J(e)(\omega), \quad \forall e \in E, \forall \omega \in \Omega_\theta.$$

Proof. Note that the conclusions of Lemmas 3.2 and 3.3 hold for $\Omega = \Delta$ and $\sigma = \text{id}_\Omega$.

(a) By Remark 3.4 and Lemma 3.2(c), we see that θ is a $C_0(\Omega)$ -module homomorphism. Furthermore, as θ_ν (as in Lemma 3.2(c)) is an orthogonality preserving (and hence bounded) linear map between Hilbert spaces for any $\nu \in T$ (where T is as in Lemma 3.2(c), with $\sigma = \text{id}_\Omega$), we know from Lemma 3.2(c) that θ is bounded (note that T is finite).

(b) By part (a), $\mathfrak{U}_\theta = \emptyset$. Thus, Lemma 3.3 tells us that there exists a bounded non-negative function ψ on Ω with $\langle \theta(e), \theta(f) \rangle = |\psi|^2 \cdot \langle e, f \rangle$. Let $\omega \in \Omega_E$ and pick any $e \in E$ such that there is $U_\omega \in \mathcal{N}_\Omega(\omega)$ with $e(\nu) \neq 0$ for all $\nu \in U_\omega$. Then $\psi(\omega) = \frac{|\theta(e)(\omega)|}{|e(\omega)|}$ for all $\omega \in U_\omega$. Hence ψ is continuous at ω , and $\varphi(\omega) = \psi(\omega)^2$ is the required function.

(c) Note that $\Omega_\theta \subseteq \Omega_E$ because of part (a). Since $\varphi(\omega) > 0$ ($\omega \in \Omega_\theta$), we know from part (b) that $\psi = \varphi^{1/2}$ gives a strictly positive element ψ_0 in $C_b(\Omega_\theta)_+$. The equivalence in [7, (2.2)] (consider E and F as Hilbert $C(\Omega_\infty)$ -bundles) tells us that the restriction of θ

induces a bounded Banach bundle map, again denoted by θ , from $\Xi^E|_{\Omega_\theta}$ into $\Xi^F|_{\Omega_\theta}$. For each $\eta \in \Xi^E|_{\Omega_\theta}$, we define $J(\eta) := \psi_0(\pi(\eta))^{-1}\theta(\eta)$ (where $\pi : \Xi^E \rightarrow \Omega$ is the canonical projection). Then $J : \Xi^E|_{\Omega_\theta} \rightarrow \Xi^F|_{\Omega_\theta}$ is a Banach bundle map (as $\eta \mapsto \psi_0(\pi(\eta))^{-1}$ is continuous) which is an isometry on each fibre (hence J is bounded) such that $\theta(\eta) = \psi(\pi(\eta))J(\eta)$. This map J induces a map, again denoted by J , in $\mathcal{B}_{C_0(\Omega_\theta)}(E_{\Omega_\theta}, F_{\Omega_\theta})$ that satisfies the requirement of part (c). \square

It is natural to ask if one can find $\varphi \in C_b(\Omega)$ such that the conclusion of Theorem 3.5(b) holds. Unfortunately, the following example tells us that it is not the case in general.

Example 3.6. Let $\Omega = \mathbb{R}_\infty$, the one-point compactification of the real line \mathbb{R} . Consider $E = C_0(\mathbb{R}) = F$ as Hilbert $C(\Omega)$ -modules and $\theta(f)(t) = f(t) \cos t$ for all $f \in E$ and $t \in \mathbb{R}$. Then $\Omega \setminus \Omega_E = \{\infty\}$ and $\varphi(t) = \cos t$ for any $t \in \mathbb{R} = \Omega_E$. Thus, one cannot extend φ to a continuous function on Ω .

Now, we can obtain the following commutative analogue of [10, 2.3]. This, together with Corollary 3.9, asserts that the orthogonality structure of a Hilbert bundle determines essentially its unitary structure, as we claimed in the Introduction. Note also that a large portion of Lemma 3.2 were used to deal with the possibility of $\theta(K_{\sigma(\nu)}^E) \not\subseteq K_\nu^F$ (such situation does not exist for $C_0(\Omega)$ -module homomorphism), and this corollary actually has a much easier proof.

Corollary 3.7. *Let Ω be a locally compact Hausdorff space, and E and F be two Hilbert $C_0(\Omega)$ -modules. Suppose that $\theta : E \rightarrow F$ is a $C_0(\Omega)$ -module homomorphism which preserves orthogonality. Then θ is bounded and there exists a bounded non-negative function φ on Ω that is continuous on Ω_E and satisfies $\langle \theta(e), \theta(f) \rangle = \varphi \cdot \langle e, f \rangle$ for all $e, f \in E$.*

Recall that a Hilbert $C_0(\Omega)$ -module E is *full* if the \mathbb{C} -linear span, $\langle E, E \rangle$, of

$$\{\langle e, f \rangle : e, f \in E\}$$

is dense in $C_0(\Omega)$.

Remark 3.8. (a) E is full if and only if $E \not\subseteq K_\omega^E$ for any $\omega \in \Omega$ (or equivalently, $\Omega_E = \Omega$). In fact, if $E \subseteq K_\omega^E$, then $f(\omega) = 0$ for any $f \in \langle E, E \rangle$ and E is not full. Conversely, if E is not full, then there exists $\omega \in \Omega$ such that $f(\omega) = 0$ for any $f \in \langle E, E \rangle$ (because the closure of $\langle E, E \rangle$ is an ideal of $C_0(\Omega)$) and $E \subseteq K_\omega^E$.

(b) If E is full, then by part (a), the function φ in Theorem 3.5(b) (and Corollary 3.7) is an element of $C_b(\Omega)$. However, there is no guarantee that this function is strictly positive.

(c) Suppose that F is full and θ is surjective orthogonality preserving local \mathbb{C} -linear map. If there exists $\omega \in \Omega \setminus \Omega_\theta$, then $F = \theta(E) \subseteq K_\omega^F$ which contradicts the fullness of F (see part (a)). Consequently, $\Omega_\theta = \Omega$. As $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$ (by Theorem 3.5(a)), we see that $\Omega = \Omega_\theta \subseteq \Omega_E$ and E is full (because of part (a)).

Corollary 3.9. *Let Ω be a locally compact Hausdorff space, and let E and F be two Hilbert $C_0(\Omega)$ -modules. Suppose that F is full and $\theta : E \rightarrow F$ is an orthogonality preserving surjective local \mathbb{C} -linear map. Then $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$. Moreover, there exist a strictly positive element $\psi \in C_b(\Omega)_+$ and a unitary $U \in \mathcal{B}_{C_0(\Omega)}(E, F)$ such that $\theta = \psi \cdot U$.*

Proof. Remark 3.8(c) tells us that $\Omega_\theta = \Omega$. By the surjectivity of θ , the bounded Banach bundle map J in Theorem 3.5 is a unitary on each fibre. Therefore, the element $U \in \mathcal{B}_{C_0(\Omega)}(E, F)$ corresponding to J as given in [7, (2.2)] is a unitary. \square

3.2. Hilbert bundles over different base spaces.

Definition 3.10. θ is said to be *quasi-local* if it is bijective and for any $e \in E$ and $\lambda \in C_0(\Delta)$, we have

$$(3.3) \quad \text{supp}_\Omega \theta^{-1}(\theta(e)\lambda) \subseteq \text{supp}_\Omega e.$$

Note that if $\Delta = \Omega$, and if θ is both local and bijective (hence θ^{-1} is also local), then θ is quasi-local by Remark 3.4.

Lemma 3.11. *Suppose that θ is bijective and quasi-local and that both θ and θ^{-1} are orthogonality preserving. Then $|\theta(e)||\theta(g)| = 0$ whenever $e, g \in E$ with $\text{supp}_\Omega e \cap \text{supp}_\Omega g = \emptyset$.*

Proof. Suppose on the contrary that there exist $e_1, e_2 \in E$ and $\nu \in \Delta$ such that $\text{supp}_\Omega e_1 \cap \text{supp}_\Omega e_2 = \emptyset$ but $\|\theta(e_1)(\nu)\| \|\theta(e_2)(\nu)\| \neq 0$. As θ is orthogonality preserving, we may assume that $\theta(e_1)(\nu)$ and $\theta(e_2)(\nu)$ are two orthogonal unit vectors in Ξ_ν^F . Let $U, W \in \mathcal{N}_\Delta(\nu)$ with $W \subseteq \text{Int}_\Delta(U)$ and $\|\theta(e_i)(\mu)\| > 1/2$ for any $\mu \in U$. Pick any $\lambda \in \mathcal{U}_\Delta(W; U)$. Define $h_i \in F \setminus \{0\}$ for $i = 1, 2$ by

$$h_i(\mu) := \begin{cases} \theta(e_i)(\mu) \frac{\lambda(\mu)}{|\theta(e_i)(\mu)|} & \mu \in \text{Int}_\Delta(U) \\ 0 & \mu \notin \text{Int}_\Delta(U) \end{cases}$$

and set $e'_i := \theta^{-1}(h_i)$. The orthogonality of h_1 and h_2 (note that e_1 and e_2 are orthogonal), together with that of $h_1 + h_2$ and $h_1 - h_2$ (as $|h_1| = \lambda = |h_2|$), ensures the orthogonality of e'_1 and e'_2 , as well as that of $e'_1 + e'_2$ and $e'_1 - e'_2$. It follows that $|e'_1| = |e'_2| \neq 0$ which contradicts the fact $|e'_1||e'_2| = 0$ (as θ is quasi-local). \square

Theorem 3.12. *Let Ω and Δ be locally compact Hausdorff spaces. Suppose that E is a full Hilbert $C_0(\Omega)$ -module and F is a full Hilbert $C_0(\Delta)$ -module. If $\theta : E \rightarrow F$ is a bijective \mathbb{C} -linear map such that both θ and θ^{-1} are quasi-local and orthogonality preserving, then θ is bounded and*

$$(3.4) \quad \theta(e)(\nu) = \psi(\nu)J_\nu(e(\sigma(\nu))), \quad \forall e \in E, \forall \nu \in \Delta,$$

where $\sigma : \Delta \rightarrow \Omega$ is a homeomorphism, ψ is a strictly positive element in $C_b(\Delta)_+$, and J_ν is a unitary operator from $\Xi_{\sigma(\nu)}^E$ onto Ξ_ν^F such that for each fixed $f \in E$, the map $\nu \mapsto J_\nu(f(\sigma(\nu)))$ is continuous.

Proof. We consider E as a Hilbert $C(\Omega_\infty)$ -module. For each $\nu \in \Delta$, let

$$S_\nu := \left\{ \omega \in \Omega_\infty : \theta(K_{\Omega_\infty \setminus W}^E) \not\subseteq K_\nu^F \text{ for every } W \in \mathcal{N}_{\Omega_\infty}(\omega) \right\}.$$

We first show that S_ν is a singleton set. Indeed, assume that $S_\nu = \emptyset$. Then for any $\omega \in \Omega_\infty$, there is $W_\omega \in \mathcal{N}_{\Omega_\infty}(\omega)$ such that $\theta(K_{\Omega_\infty \setminus W_\omega}^E) \subseteq K_\nu^F$. Consider $\omega_1, \dots, \omega_n \in \Omega_\infty$ with

$$\bigcup_{k=1}^n \text{Int}_{\Omega_\infty}(W_{\omega_k}) = \Omega_\infty,$$

and consider $\{\varphi_k\}_{k=1}^n$ to be a partition of unity subordinate to $\{\text{Int}_{\Omega_\infty}(W_{\omega_k})\}_{k=1}^n$. Then for any $e \in E$, we have $e\varphi_k \in K_{\Omega_\infty \setminus W_{\omega_k}}^E$ and so $\theta(e) \in K_\nu^F$. This shows that $F = K_\nu^F$ (as θ is surjective) which contradicts the fullness of F (see Remark 3.8(a)). Now, assume that there are distinct elements $\omega_1, \omega_2 \in S_\nu$. Let $V_1 \in \mathcal{N}_{\Omega_\infty}(\omega_1)$ and $V_2 \in \mathcal{N}_{\Omega_\infty}(\omega_2)$ with $V_1 \cap V_2 = \emptyset$. By the definition of S_ν , there exist $e_1, e_2 \in E$ with $\text{supp}_\Omega e_i \subseteq V_i \setminus \{\infty\}$ and $\theta(e_i)(\nu) \neq 0$ for $i = 1, 2$ which contradict Lemma 3.11. Thus, there is a unique element $\sigma(\nu) \in \Omega_\infty$ with $S_\nu = \{\sigma(\nu)\}$. Next, we claim that

$$(3.5) \quad \theta(I_{\sigma(\nu)}^E) \subseteq I_\nu^F, \quad \forall \nu \in \Delta.$$

Consider any $V \in \mathcal{N}_{\Omega_\infty}(\sigma(\nu))$ and $e \in K_V^E$. Pick any $U \in \mathcal{N}_{\Omega_\infty}(\sigma(\nu))$ with $U \subseteq \text{Int}_{\Omega_\infty}(V)$. By the definition of σ , there exists $g \in K_{\Omega_\infty \setminus U}^E$ such that $\theta(g)(\nu) \neq 0$. Hence, there is $W \in \mathcal{N}_\Delta(\nu)$ such that $\theta(g)(\mu) \neq 0$ for all $\mu \in W$ and Lemma 3.11 implies that $\theta(e) \in K_W^F$ as claimed. If there exists $\nu \in \Delta \setminus \Delta_\theta$, then for any $f \in F$, we have $f(\nu) = 0$ (because θ is surjective) which contradicts the fullness of F . Thus, $\Delta_\theta = \Delta$ and $\sigma : \Delta \rightarrow \Omega_\infty$ is continuous (by Lemma 3.1). As θ^{-1} is also quasi-local and orthogonality preserving, a similar argument as the above gives a continuous map $\tau : \Omega \rightarrow \Delta_\infty$ satisfying $\theta^{-1}\left(I_{\tau(\omega)}^F\right) \subseteq I_\omega^E$ for all $\omega \in \Omega$. Now, the argument of [17, Theorem 5.3] tells us that σ is a homeomorphism from Δ to Ω such that

$$\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma, \quad \forall e \in E, \forall \varphi \in C_0(\Omega),$$

and by Lemma 3.2(c), there exists a finite set T consisting of isolated points of Δ such that θ restricts to a bounded map from $K_{\sigma(T)}^E$ to K_T^F . Since any $\nu \in T$ is an isolated point, θ

induces an orthogonality preserving (hence bounded) map θ_ν from the Hilbert space $\Xi_{\sigma(\nu)}^E$ onto the Hilbert space Ξ_ν^F . This shows that θ is bounded (because of Lemma 3.2(c) and the fact that T is finite). By Lemma 3.3, there is a surjective isometry $J_\nu : \Xi_{\sigma(\nu)}^E \rightarrow \Xi_\nu^F$ such that

$$\theta(e)(\nu) = \psi(\nu)J_\nu(e(\sigma(\nu))), \quad \forall e \in E, \forall \nu \in \Delta.$$

Now the fullness of E implies that $\psi(\nu) > 0$ (for every $\nu \in \Delta$) and clearly $\nu \mapsto \frac{\theta(e)(\nu)}{\psi(\nu)}$ is continuous. \square

Note that the assumption of θ^{-1} being orthogonality preserving is necessary in Theorem 3.12 as can be seen from the following example.

Example 3.13. Let Ω be a (non-empty) locally compact Hausdorff space, and Ω_2 be the topological disjoint sum of two copies of Ω with $j_1, j_2 : \Omega \rightarrow \Omega_2$ being respectively the embeddings into the first and the second copies of Ω in Ω_2 . Let H be a (non-zero) Hilbert space, and let H_2 be the Hilbert space direct sum of two copies of H . Then the map $\theta : C_0(\Omega_2, H) \rightarrow C_0(\Omega, H_2)$ defined by

$$\theta(f)(\omega) = (f(j_1(\omega)), f(j_2(\omega)))$$

is a bijective \mathbb{C} -linear map preserving orthogonality satisfying Condition (3.3). However, θ is not of the expected form. Note that θ^{-1} does not preserve orthogonality.

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